Beauty in the defects

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based on
arXiv.21mm.nnn (with S.Jeong and N.Lee)
arXiv.21mm.nnn (with Oleksansdr Tsymbaliuk)

arXiv:2009.11199 (with Norton Lee)
arXiv:2007.03660 (with Saebyeok Jeong)
arXiv:2007.03646

and a series of BPS/CFT papers in 2015-2017

String theory dualities give us means of relating dynamics of quantum field theory in various dimensions

in the last 15 years or so a popular theme is the 4d/2d correspondence, aka AGT duality aka BPS/CFT correspondence

Today we are going to explore the physics and mathematics consequences related to defects in four dimensional gauge theory

- Knizhnik-Zamolodchikov equation in 4d gauge theory
- Painlevé VI τ-function and its generalizations

GIL (Gamayun-Iorgov-Lysovyi formula)

1207.0787

parameters

$\vec{\theta} = (\vartheta_0, \vartheta_t, \vartheta_1, \vartheta_\infty) \in \mathbb{C}^4$

$w(t)$

classical mechanics

PVI

$$\ddot{w} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)\dot{w}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)\dot{w} + \frac{2w(w-1)(w-t)}{t^2(t-1)^2}\left(\left(\theta_\infty - \frac{1}{2}\right)^2 - \frac{\theta_0^2 t}{w^2} + \frac{\theta_1^2(t-1)}{(w-1)^2} - \frac{\left(\theta_t^2 - \frac{1}{4}\right)t(t-1)}{(w-t)^2}\right)$$

two constants (initial conditions) $\mathbb{C}^2$

Hamiltonian equation + phase space

$H(p, w; t)$

τ - function

$\log \tau = S$

$$H(p, w, t) = \frac{\partial S}{\partial t}$$

$\leftarrow$ S property expressed through initial conditions

1) τ knows about the location of singularities

2) admits generalization to several degrees of freedom

$$\left(P^{2r}, \omega_{\mathbb{C}}\right)$$

$\mathbb{C}$ – sympl manifold

$k = 1, \dots, r$

$$H_k(p, q; \underbrace{t_1, \dots, t_r}_{\mathcal{T}^r})$$

$$\frac{\partial q^i}{\partial t_k} = \frac{\partial H_k}{\partial p_i}$$

$$\frac{\partial p_i}{\partial t_k} = -\frac{\partial H_k}{\partial q^i}$$

Special situation

$$\{H_k, H_\ell\}^{\omega_{\mathbb{C}}} = 0 \quad \forall k, \ell$$

$$\frac{\partial H_k}{\partial t_\ell} - \frac{\partial H_\ell}{\partial t_k} = 0$$

$$\Rightarrow \quad H_k = \frac{\partial}{\partial t_k} S$$

roughly on $P^{2r} \times \mathcal{T}^r$

PVI - isomonodromy deformation of ∇

$$\frac{\partial}{\partial t}\nabla = [\nabla, \epsilon_t]$$

$$\mathbb{C}^1 = \mathbb{CP}^1 = \overline{\mathcal{M}_{0,4}}$$

$$\mathcal{P}^2 = (\mathcal{O}_0 \times \mathcal{O}_t \times \mathcal{O}_1 \times \mathcal{O}_\infty) // SL_2(\mathbb{C})$$

$$= \left\{ \nabla = \frac{\partial}{\partial z} + \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right\}$$

$A_i \in \mathrm{Lie}\, SL_2$

$i = 0, t, 1, \infty$

$\mathrm{Tr}\, A_i^2 = 2\vartheta_i^2$

$(\vartheta_i, -\vartheta_i)$ - eigenvalues of A_i

$$\frac{d}{dt}\log\tau = \frac{dS}{dt} = \frac{\operatorname{Tr} A_0 A_t}{t} + \frac{\operatorname{Tr} A_t A_1}{t-1}$$

w is the so-called separated variable

in the gauge where $A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}$

$$A(z) = \begin{pmatrix} & \boxed{} \\ & \end{pmatrix} \qquad \# \quad \frac{z - w}{z(z-t)(z-1)}$$

location of the zero of A_{12}

important

BPZ/KZ correspondence

classical

$$\tau = e^{S} = \underbrace{\sum_{n \in \mathbb{Z}}}\ C_n(\vec{\theta}, \alpha)\, t^{(\alpha+n)^2} e^{\beta \cdot n} \times Z^{inst}(\underline{\alpha+n}, \vec{\theta}, t)$$

no $\hbar$

$(\alpha, \beta; \vec{\theta})$

quantum

AGT

conformal blocks of Liouville theory

Coulomb

instanton partition function of

$c=1$

$$t = q = \exp\left(i\theta - \frac{8\pi^2}{g^2}\right)$$

$$\alpha = \frac{a}{\hbar}$$

Ω-deformed $(\hbar, -\hbar)$

$c \to \infty$ classical

$\mathcal{N}=2$ $SU(2)$ theory $N_f = 4$

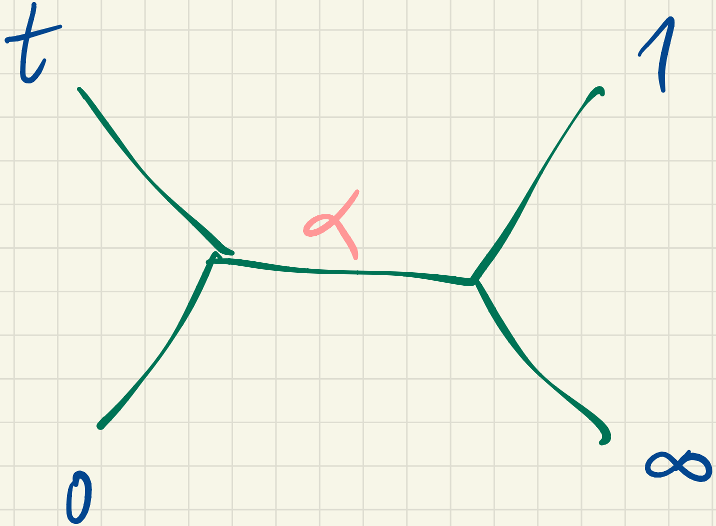
$$\Psi = e^{\frac{iS}{\hbar}}$$

$$\vec{\theta} \longleftrightarrow \frac{\vec{m}}{\hbar}$$

masses

$\beta = 0$

$\hbar \longleftrightarrow g_s$



$\beta = \frac{\partial W}{\partial \alpha}$ → eff. twisted superpot.

GIL

$\alpha \longleftrightarrow \Delta$ intermediate conf dim

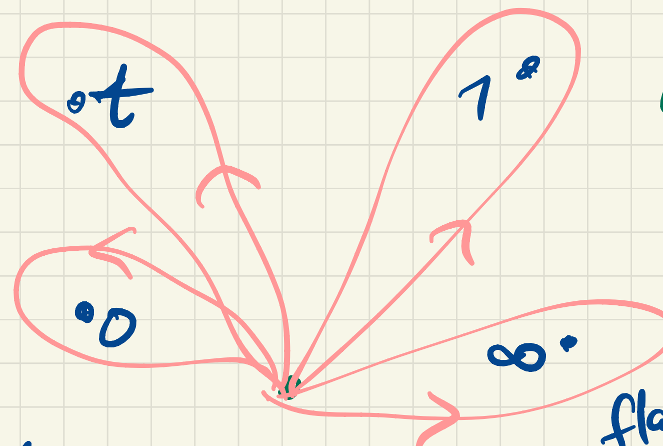
$\vec{\theta} \longleftrightarrow \Delta_0, \Delta_t, \Delta_1, \Delta_\infty$ — conf dim. of Liouville primaries

Monodromy data

$\int_\Sigma \mathrm{Tr}\, dA \wedge dA$

$d\alpha \wedge d\beta$

$\nabla \longrightarrow$



$g_0 g_t g_1 g_\infty = 1$

$= \mathcal{M}^{flat}_{SL_2(\mathbb{C})}$

$P^2 \times \{t\} \longrightarrow \{ g \mid \quad \} / G = SL_2(\mathbb{C})$

$(\widehat{sl_2})_k$ Vir_c

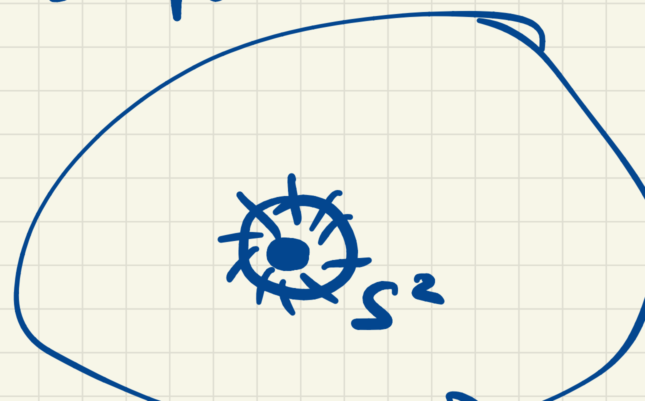
$Z_{\hat{M}^4}$

G/L — Blowup formula

related

Z_{M^4}

$\mathbb{R}^4$



$N_f = 4$ $N=2$

$Z_{\hat{\mathbb{R}}^4} = Z_{\mathbb{R}^4}$ (Nakajima-Yoshioka)

$SU(2)$

$U(1)\times U(1) \subset Spin(4)$

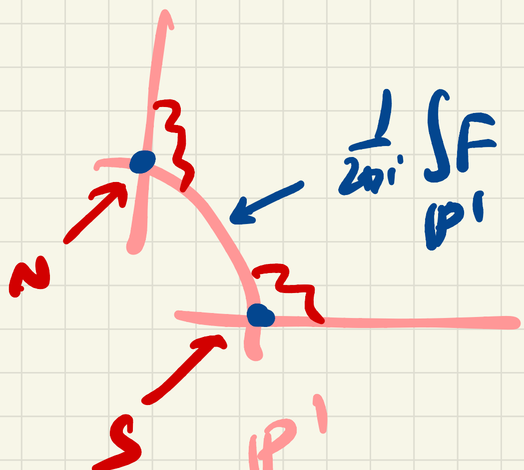
$$Z(a, \vec{m}, q;\ \varepsilon_1, \varepsilon_2) = \sum_{n\in\mathbb{Z}} Z(a+\varepsilon_1 n, \varepsilon_1, \varepsilon_2-\varepsilon_1, q)^{m} \times Z(a+\varepsilon_2 n, \varepsilon_1-\varepsilon_2, \varepsilon_2, q)^{m}$$

Ω-def

S

$N_f = 4$

$N = 2$



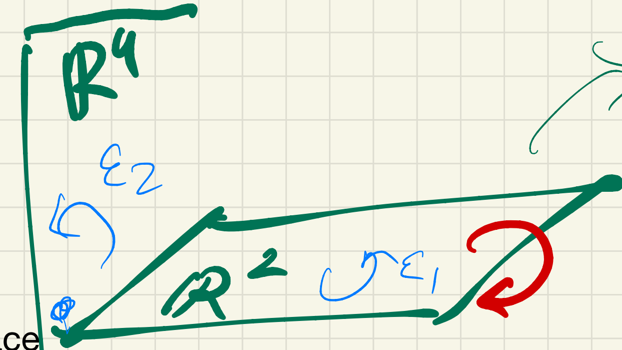
$$\frac{1}{2\pi i}\int_{\mathbb{P}^1} F = n$$

$(z_1, z_2) \in \mathbb{C}^2$

$(z_1, z_2/z_1)$ $(z_1/z_2, z_2)$

local coord

Add surface defect



depends on Kähler moduli of G/T

$P\exp \oint A$

small circle around $\mathbb{R}^2$

story of surface defects:
Kronheimer-Mrowka
Losev, Moore, NN, Shatashvili (1995)
NN(2004, lecture at Langlands meeting at IAS)
Gukov,Witten (2006-08)
Allay, Gaiotto, Gukov, Tachikawa, H.Verlinde (2009)

4d gauge theory coupled to a 2d σ-model valued in some vector bundle over $G/T = \mathbb{CP}^1$

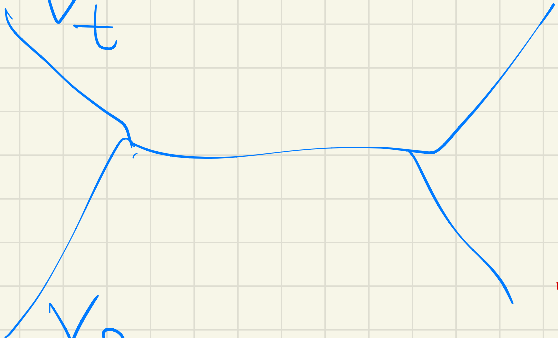
$$\Psi\left[a, z, m, \varepsilon_1, \varepsilon_2\right] \Rightarrow$$ Explicit using localization

complx Kähler moduli of 2d σ-model

Obeys Knizhnik–Zamolodchikov equation (for $SU(N)$)

(NN' 2009 conjecture)

HW V_t $\widetilde{HW}$ V_1 Kapustin's TST trick



lw Verma V_0 hw Verma V_∞

$U(N)/U(1)^N$ & Kähler moduli

Current $(\widehat{sl_N})_k$ 4-pt conformal block

$$\Psi \in \left(V_0 \otimes V_t \otimes V_1 \otimes V_\infty \right)^{sl_N} = \mathrm{Fun}(z_1, \ldots z_{N-1})$$

$$\frac{\varepsilon_2}{\varepsilon_1}\frac{d}{dq}\Psi = \left(KZ \text{ operator}\right)\Psi$$

$$\frac{\varepsilon_2}{\varepsilon_1}\frac{d}{dz_i}\Psi = \sum_{i\neq j}\frac{T_i^a\otimes T_j^a}{z_i-z_j}\Psi$$

$$q = \frac{z_2-z_1}{z_3-z_1}\,\frac{z_4-z_3}{z_4-z_2}$$

$\to k+N$

1 at q

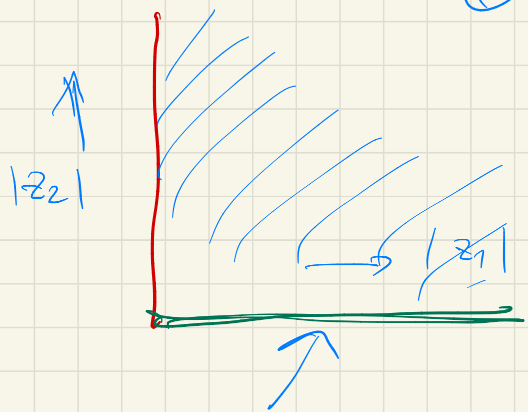
1 $\cdot$ 1

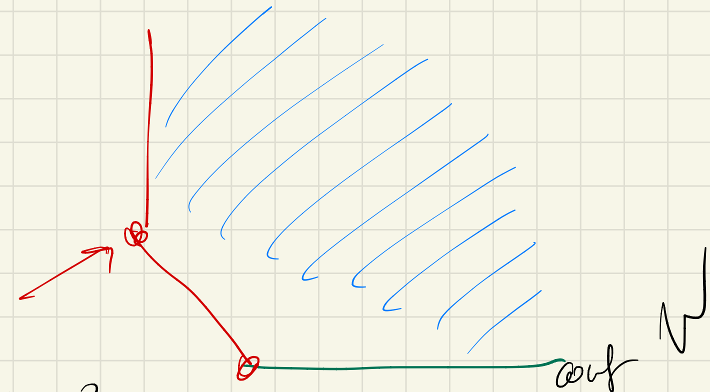
$N-1$ param at 0

$N-1$ param at ∞

GIL formula follows from blowup

$\mathbb{C}^2$ toric





surface defect at $z_2 = 0$

$c = 1 + 6Q^2$

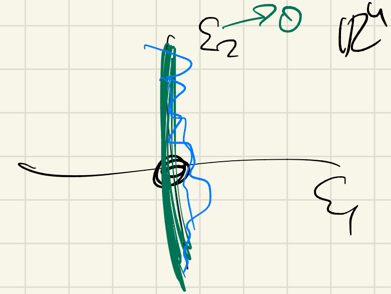
$Q = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}}$

$$\Psi_{\frac{\epsilon_2}{\epsilon_1}} = Z_C * \Psi_{\frac{\epsilon_2}{\epsilon_1 - \epsilon_2}}$$

conf block of $\widehat{sl_N}$

$$\Psi[a, z, m, q, \epsilon_1, \epsilon_2] = \sum_{n \in \mathbb{Z}} Z[a + \epsilon_1 n, \epsilon_1, \epsilon_2 - \epsilon_1]\, \Psi[a + \epsilon_2 n, \epsilon_1 - \epsilon_2, \epsilon_2]$$

$\varepsilon_2 \to 0$ limit



$\widetilde{W}_{eff}$

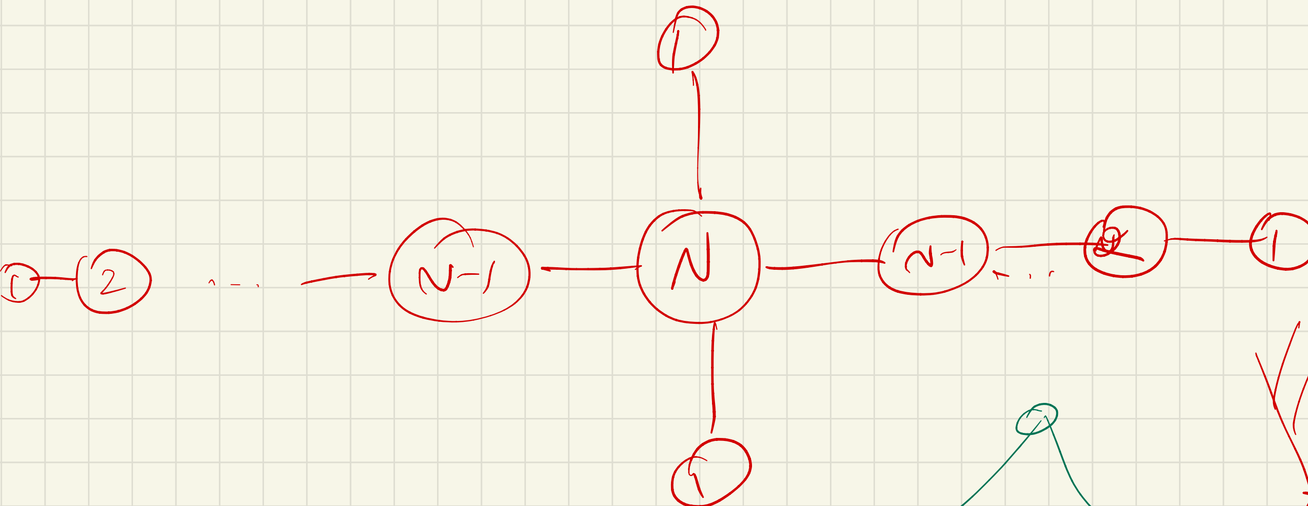
$$\Psi \sim e^{\frac{1}{\varepsilon_2} \widetilde{W}_{eff}(a, \varepsilon_1, m, q)}$$

$U(2)$

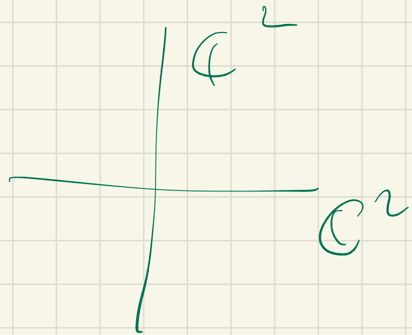
$$Z[a, \varepsilon_1, -\varepsilon_1] \times e^{\frac{1}{\varepsilon_2}\left(\widetilde{W}(a + \varepsilon_2 n, \varepsilon_1 - \varepsilon_2)\right)} = e^{\frac{\widetilde{W}}{\varepsilon_2} + n \underbrace{\frac{\partial W}{\partial a}}_{1/\beta} - \frac{\partial S}{\partial \varepsilon_2}}$$

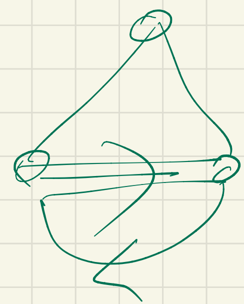
$\hookrightarrow c = 1$ Liouville conformal block W

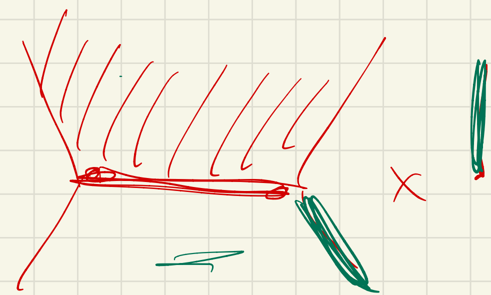
$\underline{(1-t)^{\#}}$ $\quad U(1)$ $\quad \log \tau$



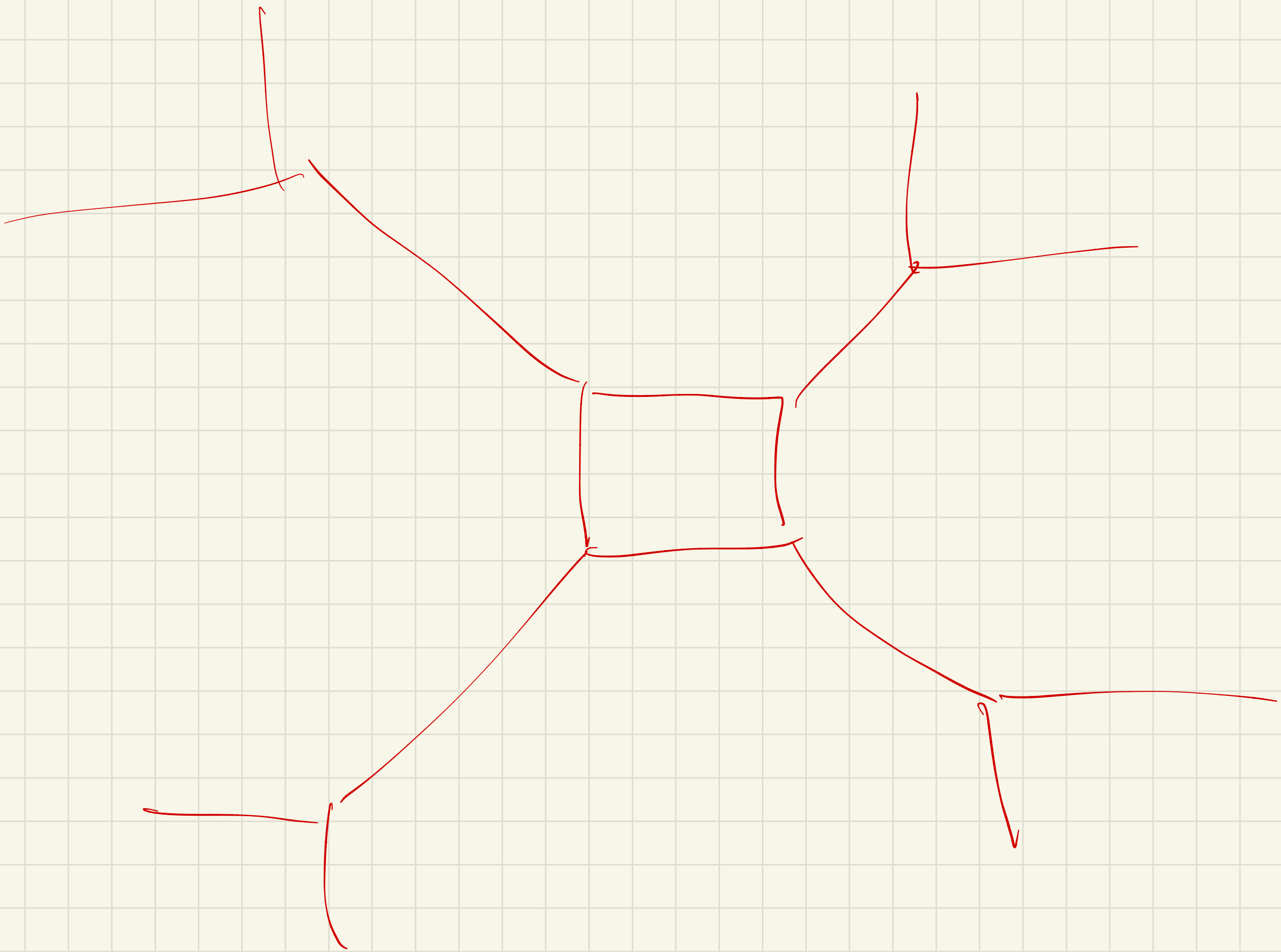
$\mathbb{C}^4$







$\rightsquigarrow$ local $\mathbb{P}^1 \times \mathbb{C}$



$r \to 0$

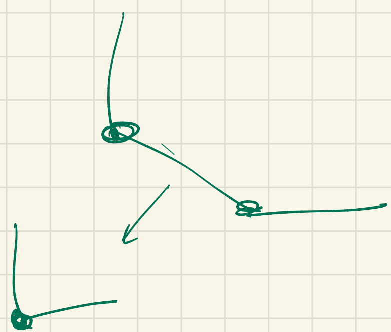
$$Z = Z + Z$$

O

$$\Psi(x) = \frac{Q(x)}{Q(x-\varepsilon_1)}$$

$$\Psi = Z + \Psi$$

$\mathbb{R}^2$



ψ

$\mathbb{R}^2$

$$\left\langle \sum_x Q(x)\, t^x \psi \right\rangle$$